Maskin Meets Abreu and Matsushima: Rationalizable Implementation*

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Abstract

We prove that the Maskin monotonicity* condition proposed by Bergemann et al. (2011) fully characterizes rationalizable implementation. Different from previous papers, our approach possesses many appealing features simultaneously, e.g., finite mechanisms (with no integer or modulo game) are used; no lottery or transfer used on the equilibrium path; the message space is smal; the implementation is robust to information perturbations and continuous in the sense of Oury and Tercieux (2012).

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1 Introduction

The design of institution to be interacted among strategic agents has been an important research agenda in economic theory. Suppose a society has decided on social choice rule – a recipe for choosing the socially optimal alternatives on the basis of individuals' preferences

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over alternatives. To tackle the problem of how to *implement* the rule, Maskin (1999), in his classic paper, (i) describes a decentralized decision making process as a *mechanism*, which specifies the possible actions available to members of a society, as well as the outcomes of these actions; and (ii) asks to what extent one can design a mechanism which makes its "all" Nash equilibrium outcomes socially desirable.[1] This is called *Nash implementation*. Maskin proposes a well-known monotonicity condition, which we refer to as *Maskin monotonicity*, and shows necessary and almost sufficient for Nash implementation.

The main purpose of our paper is to characterize the class of *social choice functions* (henceforth, SCFs) that are exactly implementable in *rationalizable strategies* by a finite mechanism. Rationalizable strategies are defined as the set of strategies that survive the iterated elimination of never best responses. In finite mechanisms, as in this paper, rationalizable strategies are equivalent to the strategies that survive the iterated elimination of strictly dominated strategies. Our mechanism excludes the *integer game* or *modulo game* constructions which are prevalent in the literature yet possess problematic feature.

We extend Nash implementation result to implementation in rationalizable strategies. In the same environments with lottery and monetary transfer, our Theorem 1 shows that an SCF is implementable in rationalizable strategies by a finite mechanism if and only if it satisfies Maskin monotonicity*, which is proposed by Bergemann et al. (2011) (henceforth, BMT) and stronger than Maskin monotonicity.[2] Theorem 1 handles the case of two agents as well as more than two agents and the mechanism constructed for Theorem 1 is simple. The sufficiency result of Bergemann et al. (2011), on the other hand, needs at least three agents and uses an infinite mechanism.

We now highlight how our results provide new insights on many classical as well as recent results in the literature. First, Oury and Tercieux (2012) advocate for rationalizable implementation by finite mechanisms. They consider the following situation: the planner wants not only that there is an equilibrium that implements the SCF but also that the same equilibrium continues to implement the SCF in all the models *close* to her initial model. Hence, the SCF is *continuously* implementable. Theorem 4 of Oury and Tercieux (2012) shows that an SCF is continuously implementable by a finite mechanism if it is exactly

[1]The original version of Maskin's paper has been circulated since 1977.

[2]BMT show that Maskin monotonicity* is a necessary rationalizable implementation using mechanisms satisfying what they called the best-response property (e.g., finite mechanisms).

implementable in rationalizable strategies by a finite mechanism.[3] What has been unknown are the conditions for exact implementation in rationalizable strategies by a finite mechanism. Our Theorem 1 fills this important gap.[4] In addition, Jain (2017) provides an example in which Maskin monotonicity* is strictly stronger than Maskin monotonicity.

We also discuss rationalizable implementation when the SCF is *responsive*, which says that any two distinct states lead to distinct outcomes. Bergemann et al. (2011) observe that when the SCF is responsive, Maskin monotonicity* reduces to Maskin monotonicity. We show that, for any SCF f, we can construct an SCF f^{ε} that is ε-close to f such that f^{ε} is responsive and satisfies Maskin monotonicity. This is summarized as our Corollary 3: "any" SCF is virtually implementable in rationalizable strategies by a finite mechanism.

The rest of the paper is organized as follows. In Section 2, we provide the basic definition and notation for this paper's setup. In Section 3, we adopt rationalizability and identify Maskin monotonicity* as a necessary and sufficient condition for rationalizable implementation by a finite mechanism. As concluding remarks, Section 4 discusses a number of extensions of our main result.

2 Preliminaries

2.1 Environment

Consider a finite set of agents $\mathcal{I} = \{1, 2, ..., I\}$ with $I \geq 2$; a finite set of possible states Θ; and a set of pure alternatives A. We consider an environment with lotteries and transfers. Specifically, we work with the space of allocations/outcomes $X \equiv \Delta(A) \times \mathbb{R}^I$ where $\Delta(A)$ denotes the set of lotteries on A that have a countable support, and $\mathbb{R}^I$ denotes the set of transfers to the agents.

Each state $\theta \in \Theta$ induces a type $\theta_i \in \Theta_i$ for each agent $i \in \mathcal{I}$. Assume that Θ has no redundancy, i.e., whenever $\theta \neq \theta'$, we must have $\theta_i \neq \theta'_i$ for some agent i. Hence, we

[3] Oury and Tercieux (2012) also prove the "only if" part of the result under a further assumption that sending messages is slightly costly.

[4] However, our Theorem 1 has two caveats in relation to Theorem 4 of Oury and Tercieux (2012): First, we focus on complete information environments, whereas Oury and Tercieux (2012) deal with general incomplete information environments. Second, we specialize in environments with lottery and monetary transfer, whereas Oury and Tercieux (2012) impose no conditions on the environments.

can identify a state θ with its induced type profile $(\theta_i)_{i\in\mathcal{I}}$ and Θ with a subset of $\times_{i=1}^{I}\Theta_i$. Moreover, we say that a type profile $(\theta_i)_{i\in\mathcal{I}}$ identifies a state θ' if $\theta_i = \theta'_i$ for every $i \in \mathcal{I}$. Each type $\theta_i \in \Theta_i$ induces a utility function $u_i(\cdot, \theta_i) : X \to \mathbb{R}$ which is quasilinear in transfers and has a bounded expected utility representation on $\Delta(A)$. That is, for each $x = \left(l, (t_i)_{i\in\mathcal{I}}\right) \in X$, we have $u_i(x, \theta_i) = v_i(l, \theta_i) + t_i$ for some bounded expected utility function $v_i(l, \theta_i)$ over $\Delta(A)$.

We focus on a *complete information* environment in which the state θ is common knowledge among the agents but unknown to a mechanism designer. Thanks to the complete-information assumption, it is indeed without loss of generality to assume that agents' values are private. The designer's objective is specified by a *social choice function* $f : \Theta \to X$, namely, if the state is θ, the designer would like to achieve the social outcome $f(\theta)$.

2.2 Mechanism and Solution

A mechanism $\mathcal{M}$ is a triplet $((M_i), g, (\tau_i))_{i\in\mathcal{I}}$ where M_i is the nonempty set of *messages* available to agent i; $g : M \to X$ $(M \equiv \times_{i=1}^{I} M_i)$ is the *outcome function*; and $\tau_i : M \to \mathbb{R}$ is the *transfer rule* which specifies the payment or subsidy to agent i. The environment and the mechanism together constitute a *game with complete information* at each state $\theta \in \Theta$ which we denote by $\Gamma(\mathcal{M}, \theta)$.

We adopt *correlated rationalizability* of Brandenburger and Dekel (1987), allowing the agents' beliefs to be correlated, as a solution concept and investigate the implications of implementation in rationalizable strategies. We define rationalizability for the finite game $\Gamma(\mathcal{M}, \theta)$ as follows. Let $S_i^0(\mathcal{M}, \theta) = M_i$, and we define $S_i^k(\mathcal{M}, \theta)$ inductively: for any $k > 0$, we set

$$S_i^k(\mathcal{M}, \theta) = \left\{ m_i \in M_i \;\middle|\; \begin{array}{l} \text{there exists } \lambda_i \in \Delta(M_{-i}) \text{ such that} \\ (1)\ \lambda_i(m_{-i}) > 0 \Rightarrow m_j \in S_j^{k-1}(\mathcal{M}, \theta) \text{ for each } j \neq i, \\ (2)\ m_i \in \arg\max_{m_i} \lambda_i(m_{-i})\, u_i(g(m_i, m_{-i}), \theta_i)\,. \end{array} \right\}.$$

Then, $S_i^\infty(\mathcal{M}, \theta) = \bigcap_{k=0}^{\infty} S_i^k(\mathcal{M}, \theta)$ is the set of rationalizable messages of agent i and $S^\infty(\mathcal{M}, \theta) = \prod_{i\in\mathcal{I}} S_i^\infty(\mathcal{M}, \theta)$ is the set of rationalizable message profiles in $\Gamma(\mathcal{M}, \theta)$.

Definition 1 *An SCF f is* ***implementable in rationalizable strategies*** *if there exists a mechanism $\mathcal{M} = ((M_i), g, (\tau_i))_{i\in\mathcal{I}}$ such that for any $\theta \in \Theta$, (i) $S^\infty(\mathcal{M}, \theta) \neq \varnothing$; and (ii)* *for any* $m \in S^{\infty}(\mathcal{M}, \theta)$, *we have* $g(m) = f(\theta)$ *and* $\tau_i(m) = 0$.

Remark: Since we propose a finite implementing mechanism below, $S^{\infty}(\mathcal{M}, \theta)$ is always nonempty, namely, requirement (i) of rationalizable implementation is automatically satisfied.

2.3 Dictator Lottery

First, we state an assumption which we impose by following Abreu and Matsushima (1992, 1994). Recall that $v_i(\cdot, \theta_i)$ denotes the bounded expected utility function of agent i of type θ_i.

Assumption 1 *For each agent* i*, we assume (i) for any type* θ_i*, there are alternatives* a *and* a' *in* A *such that* $v_i(a, \theta_i) \neq v_i(a, \theta_i)$*; (ii)* $\theta_i \neq \theta_i'$ *implies that* $v_i(\cdot, \theta_i)$ *and* $v_i(\cdot, \theta_i')$ *induce different preference orders on* $\Delta(A)$.

Let $\tilde{A} \equiv A \cup f(\Theta) \cup \cup_{i \in \mathcal{I}, \theta_i \in \Theta_i, \tilde{\theta} \in \Theta} B_{\theta_i}(\tilde{\theta})$. Since $v_i(\cdot, \theta_i)$ is bounded and Θ is finite, we choose $\eta' > 0$ as an upper bound on the monetary value of a change in the selection of an alternative in $\tilde{A}$, that is,

$$\eta' > \sup_{i \in \mathcal{I}, \theta_i \in \Theta_i, x, x' \in \tilde{A}} u_i(x, \theta_i) - u_i(x', \theta_i), \tag{1}$$

where we abuse notation to identify A with a subset of X, i.e., each $a \in A$ is identified with $x^a = (a, 0, ..., 0) \in X$.

Given this assumption, we have the following lemma.

Lemma 1 *Suppose that Assumption 1 holds. Then, for each agent* $i \in \mathcal{I}$*, there exists a function* $y_i : \Theta_i \to X$ *such that for any types* θ_i *and* θ_i' *of agent* i *with* $\theta_i \neq \theta_i'$*, we have*

$$u_i(y_i(\theta_i), \theta_i) > u_i(y_i(\theta_i'), \theta_i); \tag{2}$$

moreover, for each type θ_j' *of agent* $j \in \mathcal{I}$*, we also have for any* $x \in f(\Theta)$

$$u_i(y_j(\theta_j'), \theta_i) < u_i(x, \theta_i). \tag{3}$$

The existence of lotteries $\{y_i'(\theta_i)\} \subset \Delta(A)$ which satisfy condition (2) is proved in Abreu and Matsushima (1992). To satisfy condition (3), we simply add a penalty of η' to each outcome of the lotteries $\{y_i'(\theta_i)\}$. We call the resulting lotteries the *dictator lotteries* for agent i and denote them by $\{y_i(\theta_i)\}$.

2.3.1 Maskin Monotonicity*

In this section, we introduce a central condition to our rationalizable implementation result, which is called *Maskin monotonicity**. The condition is proposed by Bergemann et al. (2011) as a necessary condition for rationalizable implementation using "well behaved" (such as finite) mechanisms.

For $(\theta_i, x) \in \Theta_i \times X$, recall that $\mathcal{SL}_i(x, \theta_i)$ denote the strict lower-contour set at lottery x for type θ_i. For a given SCF f, we let $\mathcal{P}_f = \{\Theta_z\}_{z \in f(\Theta)}$ be the partition on Θ induced by f where $\Theta_z = \{\theta \in \Theta | \ f(\theta) = z\}$. For each partition $\mathcal{P}$ on Θ, we denote by $\mathcal{P}(\theta)$ the atom in $\mathcal{P}$ which contains state θ and $\mathcal{P}_i(\theta)$ be the projection of $\mathcal{P}(\theta)$ on Θ_i. Moreover, for each $x \in X$, let

$$\mathcal{SL}_i(x, \mathcal{P}(\theta)) \equiv \bigcap_{\hat{\theta} \in \mathcal{P}(\theta)} \mathcal{SL}_i(x, \hat{\theta}_i).$$

The following definition is obtained by adapting Definition 5 of Bergemann et al. (2011) to our setup that accommodates both lotteries and transfers.

Definition 2 *Say an SCF f satisfies* ***Maskin monotonicity**** *if there exists a partition $\mathcal{P}$ of Θ such that (i) $\mathcal{P}$ is at least as fine as $\mathcal{P}_f$; (ii) for any $\tilde{\theta}, \theta \in \Theta$, whenever $\tilde{\theta} \notin \mathcal{P}(\theta)$, there exists $i \in \mathcal{I}$ for whom*

$$\mathcal{L}_i(f(\tilde{\theta}), \mathcal{P}(\tilde{\theta})) \cap \mathcal{SU}_i(f(\tilde{\theta}), \theta_i) \neq \varnothing. \tag{4}$$

Although Maskin monotonicity* implies Maskin monotonicity, it was not a priori clear whether the two conditions are indeed different. Jain (2017) has recently constructed an example showing that strict Maskin monotonicity* is strictly stronger than Maskin monotonicity. In Section 5, we modify Jain's example to make the same point in our setup, which accommodates the case with two agents and lotteries and transfers. Since Maskin monotonicity* is a necessary condition for rationalizable implementation by a finite mechanism, we conclude that rationalizable implementation is generally more restrictive than Nash implementation, even when we focus on finite mechanisms and allow for lotteries and transfers.[5]

[5]This exhibits a contrast with Kunimoto and Serrano (2019), who argue that rationalizable implementation can be more permissive than Nash implementation when we consider social choice correspondences and allow for arbitrary infinite mechanisms.

As we introduce strict Maskin monotonicity, we say that an SCF f satisfies strict Maskin monotonicity* if we replace $\mathcal{L}_i(f(\tilde{\theta}), \mathcal{P}(\tilde{\theta}))$ in (4) with $\mathcal{SL}_i(f(\tilde{\theta}), \mathcal{P}(\tilde{\theta}))$. Again, in the environment with transfers, strict Maskin monotonicity* and Maskin monotonicity* are equivalent conditions.

Let $\mathcal{P}$ be the partition in the definition of strict Maksin monotonicity*. As the case of Nash implementation, we also make use of the best challenge scheme with respect to $\mathcal{P}$. Fix agent i of type θ_i. For each state $\tilde{\theta} \in \Theta$, if $\mathcal{SL}_i(f(\tilde{\theta}), \mathcal{P}(\tilde{\theta})) \cap \mathcal{SU}_i(f(\tilde{\theta}), \theta_i) \neq \varnothing$, we select some $x(\tilde{\theta}, \theta_i) \in \mathcal{SL}_i(f(\tilde{\theta}), \mathcal{P}(\tilde{\theta})) \cap \mathcal{SU}_i(f(\tilde{\theta}), \theta_i)$. The best challenge scheme for agent i of type θ_i with respect to $\mathcal{P}$ is defined as a function $B_{\theta_i} : \Theta \to X$ such that for any $\tilde{\theta} \in \Theta$,

$$B_{\theta_i}(\tilde{\theta}) = \begin{cases} f(\tilde{\theta}), & \text{if } \mathcal{SL}_i(f(\tilde{\theta}), \mathcal{P}(\tilde{\theta})) \cap \mathcal{SU}_i(f(\tilde{\theta}), \theta_i) = \varnothing; \\ x(\tilde{\theta}, \theta_i), & \text{if } \mathcal{SL}_i(f(\tilde{\theta}), \mathcal{P}(\tilde{\theta})) \cap \mathcal{SU}_i(f(\tilde{\theta}), \theta_i) \neq \varnothing \end{cases}$$

where we omit the reference to $\mathcal{P}$ in B_{θ_i} for simplicity.

The following lemma shows that there is a challenge scheme under which truth-telling induces the best allocation.

Lemma 2 *There is a challenge scheme $\{B_{\theta_i}(\tilde{\theta})\}$ for an SCF f such that for any state $\tilde{\theta}$ and type θ_i,*

$$u_i(B_{\theta_i}(\tilde{\theta}), \theta_i) \geq u_i(B_{\theta_i'}(\tilde{\theta}), \theta_i), \forall \theta_i' \in \Theta_i. \tag{5}$$

Proof. Fix an arbitrary challenge scheme $\{B_{\theta_i}(\tilde{\theta})\}$ for the SCF f. Without loss of generality, we may assume that for each state $\tilde{\theta}$ and each pair of types θ_i and θ_i'

$$B_{\theta_i}(\tilde{\theta}) \neq f(\tilde{\theta}) \text{ and } B_{\theta_i'}(\tilde{\theta}) \neq f(\tilde{\theta}) \Rightarrow u_i(B_{\theta_i}(\tilde{\theta}), \theta_i) \geq u_i(B_{\theta_i'}(\tilde{\theta}), \theta_i). \tag{6}$$

Indeed, if (6) does not hold for $\{B_{\theta_i}(\tilde{\theta})\}$, then whenever $B_{\theta_i}(\tilde{\theta}) \neq f(\tilde{\theta})$, we redefine $B_{\theta_i}(\tilde{\theta})$ as the most preferred allocation of type θ_i in the set $\left\{B_{\theta_i'}(\tilde{\theta}) : \theta_i' \in \Theta_i \text{ and } B_{\theta_i'}(\tilde{\theta}) \neq f(\tilde{\theta})\right\}$. It is straightforward to see that $\{B_{\theta_i}(\tilde{\theta})\}$ remains a challenge scheme with this modification.

Next, for each state $\tilde{\theta}$ and type θ_i, we show that $\{B_{\theta_i}(\tilde{\theta})\}$ satisfies (5). We proceed by considering the following two cases. First, suppose that $B_{\theta_i}(\tilde{\theta}) \neq f(\tilde{\theta})$. Then, by (6), it suffices to consider type θ_i' with $B_{\theta_i'}(\tilde{\theta}) = f(\tilde{\theta})$. Indeed, if $B_{\theta_i'}(\tilde{\theta}) = f(\tilde{\theta})$ and $B_{\theta_i}(\tilde{\theta}) \neq f(\tilde{\theta})$, then it follows from $B_{\theta_i}(\tilde{\theta}) \in \mathcal{SU}_i(f(\tilde{\theta}), \theta_i)$ that $u_i(B_{\theta_i}(\tilde{\theta}), \theta_i) > u_i(B_{\theta_i'}(\tilde{\theta}), \theta_i)$. Hence, (5) holds. Second, suppose that $B_{\theta_i}(\tilde{\theta}) = f(\tilde{\theta})$. Then, it suffices to consider type θ_i' with $B_{\theta_i'}(\tilde{\theta}) \neq f(\tilde{\theta})$. Since $B_{\theta_i}(\tilde{\theta}) = f(\tilde{\theta})$, we have $\mathcal{SL}_i(f(\tilde{\theta}), \mathcal{P}(\tilde{\theta})) \cap \mathcal{SU}_i(f(\tilde{\theta}), \theta_i) = \varnothing$. Moreover, $B_{\theta_i'}(\tilde{\theta}) \neq$

$f(\tilde{\theta})$ implies that $B_{\theta_i'}(\tilde{\theta}) \in \mathcal{SL}_i(f(\tilde{\theta}), \mathcal{P}(\tilde{\theta}))$. Hence, we must have $B_{\theta_i'}(\tilde{\theta}) \notin \mathcal{SU}_i(f(\tilde{\theta}), \theta_i)$. That is, (5) also holds. ■

In the following, we shall invoke a challenge scheme which satisfies (5) and we call it the *best challenge scheme*.

We introduce the following definition.

Definition 3 *Say that a partition $\mathcal{P}$ on states has **product structure** if, for every state $\theta \in \Theta$, $\mathcal{P}(\theta)$ is a product subset of Θ, i.e., $\mathcal{P}(\theta) = \times_{i \in \mathcal{I}} \mathcal{P}_i(\theta)$ with $\mathcal{P}_i(\theta) \subset \Theta_i$.*

For Theorem 1 which we will state and prove below, it entails no loss of generality to assume that the partition $\mathcal{P}$ has product structure. Indeed, if the SCF f satisfies strict Maskin monotonicity* on Θ, we can extend f and $\mathcal{P}$ to some new state space Θ' so that we can redefine the SCF $f' : \Theta' \to \Delta(A)$ and the partition $\mathcal{P}'$ on Θ' in such a way that f' still satisfies strict Maskin monotonicity* on Θ' and $\mathcal{P}'$ has product structure. Moreover, Assumption 1 still holds on Θ' so long as it holds on Θ.[6]

3 Rationalizable Implementation

We now state our main result on rationalizable implementation as follows.

Theorem 1 *An SCF f is implementable in rationalizable strategies by a finite mechanism if and only if it satisfies Maskin monotonicity*.*

Since a finite mechanism satisfies the *best response property* defined in Bergemann et al. (2011) (see Definition 6 of Bergemann et al. (2011) for its precise definition), the "only if" part of Theorem 1 follows from Proposition 3 of Bergemann et al. (2011). In the following subsections, we will construct a mechanism to prove the "if" part of Theorem 1.

[6]For each $P \in \mathcal{P}$, define $P_i = P$ for each $i \in \mathcal{I}$ and $\Theta^P \equiv \times_{i \in \mathcal{I}} P_i$. Let $\Theta' \equiv \bigsqcup_{P \in \mathcal{P}} \Theta^P$ as the disjoint union of Θ^P over $P \in \mathcal{P}$. By construction, Θ' has product structure. Then, for each $\theta' \in \Theta^P \subset \Theta'$, define $f'(\theta')$ as the common outcome of f in P (recall that $\mathcal{P}$ is finer than $\mathcal{P}_f$). It is easy to show that the translated SCF $f' : \Theta' \to \Delta(A)$ satisfies strict maskin monotonicity* as long as the original SCF $f : \Theta \to \Delta(A)$ does. Finally, we note that Assumption 1 holds on Θ' as long as it holds on Θ. Thus, we can translate any state space Θ into a product state space Θ' so that everything carries over to the translated state space.

3.1 The Mechanism

3.1.1 Message Space:

A generic message of agent i is:

$$m_i = \left(m_i^1, m_i^2\right) \in M_i^1 \times M_i^2 = M_i = \Theta_i \times \left[\times_{j=1}^{I} \Theta_j\right].$$

That is, agent i is asked to make (1) an announcements of his own type (which we denote by m_i^1); and (2) an announcement of a type profile (which we denote by m_i^2). To simplify the notation, we write $m_{i,j}^2 = \tilde{\theta}_j$ if agent i reports in m_i^2 that agent j is of type $\tilde{\theta}_j$.

3.1.2 Allocation Rule:

Say two states θ and θ' are equivalent (denoted as $\theta \sim \theta'$) if they belong to the same atom of $\mathcal{P}$. We further denote by $\mathcal{P}_i\left(\Theta_i\right)$ as the projection of $\mathcal{P}_i\left(\Theta\right)$ on agent i's types. Thanks to the product structure, $\mathcal{P}_i\left(\Theta\right)$ is also a partition over agent i's types. Thus, we say two types θ_i and θ_i' are equivalent (denoted as $\theta_i \sim_i \theta_i'$) if they belong to the same atom of $\mathcal{P}_i\left(\Theta_i\right)$.

Given a message profile m, we say that m is *consistent* if there exists $\tilde{\theta} \in \Theta$ such that

$$m^1 \text{ identifies } \tilde{\theta} \text{ and } m_i^2 \ \sim \ \tilde{\theta} \text{ for any } i \in \mathcal{I}. \tag{7}$$

That is, consistency requires that the type profile m^1 identifies a state $\tilde{\theta}$ that is equivalent to the state identified by m_i^2 for any $i \in \mathcal{I}$. Note that m is consistent implies that $B_{m_i^1}\left(m_i^2\right) = f\left(m_i^2\right)$ for all $i \in \mathcal{I}$.

For each message profile $m \in M$, the allocation is defined as follows:

$$g\left(m\right) = \frac{1}{I}\sum_{i\in\mathcal{I}}\left[e\left(m\right) y_i\left(m_i^1\right) \oplus \left(1 - e\left(m\right)\right) B_{m_i^1}\left(m_i^2\right)\right]$$

where $y_k : \Theta \to X$ is the dictator lottery for agent k defined in Lemma 1; moreover, we define

$$e(m) = \begin{cases} 0, & \text{if } m \text{ is consistent;} \\ \varepsilon, & \text{if } m \text{ is inconsistent and for every } i,\ m_i^2 \text{ identifies a state;} \\ 1 & \text{if } m \text{ is inconsistent and for some } i,\ m_i^2 \text{ does not identify a state.} \end{cases}$$

That is, the designer first chooses each agent i with equal probability, to use agent i's first report to check i's second report in determining the allocation.

In words, the outcome function distinguishes three cases: (1) if $e(m) = 0$, then we implement $f(m_i^2)$; (2) if $e(m) = \varepsilon$, we implement the compound lottery:

$$C_{i,i}^{\varepsilon}(m) \equiv \varepsilon \times y_i\left(m_i^1\right) \oplus (1-\varepsilon) \times B_{m_i^1}\left(m_i^2\right),$$

That is, with probability ε, we select an agent k and implement the lottery $y_k(\cdot)$ according to his first report; (3) if $e(m) = 1$, we implement the lottery $y_i(m_i^1)$.

By (1), we can choose $\varepsilon > 0$ sufficiently small, and $\eta > 0$ sufficient large such that (i) we have

$$\eta > \sup_{i \in \mathcal{I}, \theta_i \in \Theta_i, m, m' \in M} u_i(g(m), \theta_i) - u_i(g(m'), \theta_i); \tag{8}$$

(ii) it does not disturb the "effectiveness" of agent i's challenge, i.e.,

$$\begin{array}{l} B_{m_i^1}(m_i^2) \neq f(m_i^2) \Rightarrow \text{for any } \tilde{\theta} \in \mathcal{P}(m_i^2), \\ \frac{1}{I}(1-\varepsilon)\left(u_i(B_{m_i^1}(\tilde{\theta}), m_j^1) - u_i(f(\tilde{\theta}), m_i^1)\right) - \varepsilon\eta > 0 \end{array} \tag{9}$$

In other words, whenever agent i is called, it is strictly better for agent i to tell the truth in m_i^1 if therefore agent i tells a lie in m_i^2. Although agent i is picked with probability $\frac{1}{I}$, the gain is large enough when we choose ε small enough to mitigate the effect from the dictator lotteries.

3.1.3 Transfer Rule:

We now define the transfer rule. For any message profile m and any agent i, we specify the transfer to agent i as follows:

$$\tau_i(m) = \tau_i^1(m) + \tau_i^2(m),$$

where

$$\tau_i^1(m) = \begin{cases} 0, & \text{if } m_{i,j}^2 \sim_j m_j^1 \text{ for all } j \neq i; \\ -\eta, & \text{otherwise.} \end{cases} \tag{10}$$

$$\tau_i^2(m) = \begin{cases} 0, & \text{if } m_{i,i}^2 \sim_i m_{j,i}^2 \text{ for all } j \neq i; \\ -\eta, & \text{otherwise.} \end{cases} \tag{11}$$

In words, $\tau_i^1(m)$ cross-checks whether agent i's second report about agent j's type is equivalent to agent j's first report; and $\tau_i^3(m)$ cross-checks whether agent i's second report about his own type is equivalent to agent j's second report about agent i's type.

We start by outlining the proof of Theorem 1. In this proof, we first argue that if m is rationalizable, then m^1 identifies a state which is equivalent to the true state. Next, the cross-checking transfers ensure that m^2 also identifies a state equivalent to the true state. The lack of correct belief in rationalizability, as described in the above example, also necessitates that agent i must have an opportunity to self-challenge his own state announcement. Otherwise, agent i may report a state that is outside the equivalence class of the true state while believing that the lie will not be challenged by any other agent.

3.2 Proof of Theorem 1

Let $\theta \in \Theta$ be a true state. We prove the "if" part of Theorem 1 in three steps.

Step 1: *For any agent $i \in \mathcal{I}$ and any $m_i \in S_i^{\infty}(\mathcal{M}, \theta)$, we have $m_i^1 \sim_i \theta_i$.*

Fix $i \in \mathcal{I}$ and $m_i \in S_i^{\infty}(\mathcal{M}, \theta)$. Then, there is a conjecture $\lambda_i \in \Delta\left(S_{-i}^{\infty}(\mathcal{M}, \theta)\right)$ against which m_i is a best reply. Note that for any m, agent i's 1st report m_i^1 has no effect on his transfer, but only affects his payoff either through the dictator lottery when $e(m_i, m_{-i}) = 1$ or through the dictator lottery and $B_{m_i^1}(m_i^2)$ when $e(m_i, m_{-i}) = \varepsilon$ (in particular, agent i is chosen to be checked). We now show $m_i^1 \sim_i \theta_i$ by consider two cases:

Case 1. $e(m_i, m_{-i}) \neq 0$ for any m_{-i} with $\lambda_i(m_{-i}) > 0$. Consider the subcase that $e(m_i, m_{-i}) = \varepsilon$. Suppose that $m_i^1 \nsim_i \theta_i$. Let $\tilde{m}_i = (\theta_i, m_i^2)$. We show that $\tilde{m}_i$ is strictly better than m_i against λ_i in each of the following two subcases, which leads to a contradiction. Note that $e(\tilde{m}_i, m_{-i}) \neq 1$. Firstly, we consider the subcase that $e(\tilde{m}_i, m_{-i}) = \varepsilon$. Thus, $g(m_i, m_{-i})$ differs from $g(\tilde{m}_i, m_{-i})$ only when agent i is chosen to be checked, and the difference only lies in the dictator lotteries and $B_{(\cdot)}(m_i^2)$. By (2) of Lemma 1 and Lemma 2, $\tilde{m}_i$ is strictly better than m_i. Secondly, we consider the subcase that $e(\tilde{m}_i, m_{-i}) = 0$. That is, $(\tilde{m}_i, m_{-i})$ triggers neither challenge nor inconsistency. For any pair of agents chosen, the outcome involves no dictator lotteries. Hence,

$$g(\tilde{m}_i, m_{-i}) = \frac{1}{I} \sum_{i \in \mathcal{I}} B_{m_i^1}\left(m_i^2\right).$$

By (3) of Lemma 1, $\tilde{m}_i$ is strictly better than m_i.

Consider the subcase that $e(m_i, m_{-i}) = 1$. Suppose that $m_i^1 \nsim_i \theta_i$. Let $\tilde{m}_i = (\theta_i, m_i^2)$. We show that $\tilde{m}_i$ is strictly better than m_i against λ_i in each of the following two subcases, which leads to a contradiction. Note that $e(\tilde{m}_i, m_{-i})$ can only be 1. Thus, $g(m_i, m_{-i})$ differs from $g(\tilde{m}_i, m_{-i})$ only when agent i is chosen to be checked, and the difference only lies in the dictator lotterie. By (2) of Lemma 1, $\tilde{m}_i$ is strictly better than m_i.

Hence, we obtain $m_i^1 \sim_i \theta_i$ in Case 1.

Case 2. $e(m_i, m_{-i}) = 0$ for some m_{-i} with $\lambda_i(m_{-i}) > 0$. That is, there exists $\tilde{\theta} \in \Theta$ such that

$$\left(m_i^1, m_{-i}^1\right) \text{ identifies } \tilde{\theta};$$
$$m_i^2 \sim \tilde{\theta} \text{ and } B_{m_i^1}(\tilde{\theta}) = f(\tilde{\theta}), \forall i \in \mathcal{I}.$$

We claim that $\tilde{\theta} \in \mathcal{P}(\theta)$ which implies $m_i^1 \sim_i \theta_i$. Suppose on the contrary that $\tilde{\theta} \notin \mathcal{P}(\theta)$. Then, since f satisfies strict Maskin monotonicity*, there exists some agent $j \in \mathcal{I}$ for whom $B_{\theta_j}(\tilde{\theta}) \neq f(\tilde{\theta})$, and

$$u_j(B_{\theta_j}(\tilde{\theta}), \theta_j) > u_j(f(\tilde{\theta}), \theta_j). \tag{12}$$

Now we construct $\tilde{m}_j = \left(\theta_j, m_j^2\right)$. In the following, we shall show that $\tilde{m}_j = \left(\theta_j, m_j^2\right)$ strictly dominates m_j, which contradicts the hypothesis that $m_j \in S_j^\infty(\mathcal{M}, \theta)$.

Fix $\tilde{m}_{-j} \in S_{-j}^\infty(\mathcal{M}, \theta)$. Observe first that $e(\tilde{m}_j, \tilde{m}_{-j}) = \varepsilon$ because we have $\tilde{m}_j^2 \sim \tilde{\theta}$ and $B_{\theta_j}(\tilde{\theta}) \neq f(\tilde{\theta})$. Thus,

$$\begin{aligned} g(\tilde{m}_j, \tilde{m}_{-j}) &= \frac{1}{I}\sum_{k \neq j}\left[\varepsilon y_k\left(\tilde{m}_k^1\right) \oplus (1-\varepsilon) B_{\tilde{m}_k^1}\left(\tilde{m}_k^2\right)\right] \\ &\oplus \frac{1}{I}\left[\varepsilon y_j(\tilde{m}_j^1) \oplus (1-\varepsilon) B_{\tilde{m}_j^1}(m_j^2)\right] \end{aligned}$$

where $B_{\tilde{m}_j^1}(m_j^2) = B_{\theta_j}(\tilde{\theta}) \neq f(m_j^2)$. In contrast,

$$\begin{aligned} g(m_j, \tilde{m}_{-j}) &= \frac{1}{I}\sum_{k \neq j}\left[e(m_j, \tilde{m}_{-j}) y_k\left(\tilde{m}_k^1\right) \oplus (1 - e(m_j, \tilde{m}_{-j})) B_{\tilde{m}_k^1}\left(\tilde{m}_k^2\right)\right] \\ &\oplus \frac{1}{I}\left[e(m_j, \tilde{m}_{-j}) y_j\left(m_j^1\right) \oplus (1 - e(m_j, \tilde{m}_{-j})) B_{m_j^1}\left(m_j^2\right)\right] \end{aligned}$$

where $B_{m_j^1}\left(m_j^2\right) = f(m_j^2)$.

First, if $e(m_j, \tilde{m}_{-j}) = \varepsilon$, then

$$\begin{aligned} g(m_j, \tilde{m}_{-j}) &= \frac{1}{I}\sum_{k \neq j}\left[\varepsilon y_k\left(\tilde{m}_k^1\right) \oplus (1-\varepsilon) B_{\tilde{m}_k^1}\left(\tilde{m}_k^2\right)\right] \\ &\oplus \frac{1}{I}\left[\varepsilon y_j\left(m_j^1\right) \oplus (1-\varepsilon) B_{m_j^1}\left(m_j^2\right)\right]. \end{aligned}$$

Thus, $g(\tilde{m}_j, \tilde{m}_{-j})$ differs from $g(m_j, \tilde{m}_{-j})$ only when agent j is chosen to be checked. In terms of dictator lotteries, by Lemma 1, $\tilde{m}_j$ is strictly better than m_j; moreover, in terms of allocations from best challenge schemes $B_{(\cdot)}\left(m_j^2\right)$, $\tilde{m}_j$ is strictly better than m_j by (12). Hence, in this case, $\tilde{m}_j$ strictly dominates m_j.

Second, if $e(m_j, \tilde{m}_{-j}) = 1$, then

$$g(m_j, \tilde{m}_{-j}) = \frac{1}{I} \sum_{k \neq j} y_k\left(\tilde{m}_k^1\right) \oplus \frac{1}{I} y_j\left(m_j^1\right).$$

Thus, by Lemma 1, $\tilde{m}_j$ is strictly better than m_j. Hence, in this case, $\tilde{m}_j$ strictly dominates m_j.

Third, if $e(m_j, \tilde{m}_{-j}) = 0$, that is, $(m_j, \tilde{m}_{-j})$ is consistent, $B_{\tilde{m}_k^1}(\tilde{m}_k^2) = f(\tilde{m}_k^2)$ for any $k \in \mathcal{I}$, then

$$\begin{aligned} g(m_j, \tilde{m}_{-j}) &= \frac{1}{I} \sum_{k \neq j} f\left(\tilde{m}_k^2\right) \oplus \frac{1}{I} f\left(\tilde{m}_k^2\right) \\ &= \frac{1}{I} \sum_{k \neq j} f\left(\tilde{m}_k^2\right) \oplus \frac{1}{I} f(\tilde{\theta}), \end{aligned}$$

and

$$\begin{aligned} g(\tilde{m}_j, \tilde{m}_{-j}) &= \frac{1}{I} \sum_{k \neq j} \left[\varepsilon y_k\left(\tilde{m}_k^1\right) \oplus (1-\varepsilon) f\left(\tilde{m}_k^2\right)\right] \\ &\quad \oplus \frac{1}{I}\left[\varepsilon y_j(\tilde{m}_j^1) \oplus (1-\varepsilon) B_{\theta_j}(\tilde{\theta})\right] \\ &= \varepsilon\left[\frac{1}{I} \sum_{k \neq j} y_k\left(\tilde{m}_k^1\right) \oplus \frac{1}{I} y_j(\tilde{m}_j^1)\right] \\ &\quad \oplus (1-\varepsilon)\left[\frac{1}{I} \sum_{k \neq j} f\left(\tilde{m}_k^2\right) \oplus \frac{1}{I} B_{\theta_j}(\tilde{\theta})\right] \end{aligned}$$

We compare the payoff difference from choosing $\tilde{m}_j$ rather than m_j in the following way. With probability ε, the loss from choosing $\tilde{m}_j$ rather than m_j is bounded below by $-\eta$; with probability $(1-\varepsilon)$, the gain from $\tilde{m}_j$ rather than m_j is

$$\frac{1}{I}\left(u_j(B_{\theta_j}(\tilde{\theta}), \theta_j) - u_j(f(\tilde{\theta}), \theta_j)\right).$$

In total, the gain is at least

$$-\varepsilon\eta + (1-\varepsilon)\frac{1}{I}\left(u_j(B_{\theta_j}(\tilde{\theta}), \theta_j) - u_j(f(\tilde{\theta}), \theta_j)\right) > 0,$$

which follows from (9). Thus, we conclude a contradiction to the hypothesis that $m_j \in S_j^{\infty}(\mathcal{M}, \theta)$. This completes the proof of Step 1.

Step 2: *For any agent* $i \in \mathcal{I}$ *and any* $m_i \in S_i^{\infty}(\mathcal{M}, \theta)$, *we have* $m_i^2 \sim \tilde{\theta}$ where $\tilde{\theta} \in \mathcal{P}(\theta)$.

By Step 1, we know that for every $i \in \mathcal{I}$, if $m_i \in S_i^{\infty}(\mathcal{M}, \theta)$, then there exists $\hat{\theta}_{-i}$ such that $(m_i^1, \hat{\theta}_{-i})$ identifies some $\tilde{\theta} \in \mathcal{P}(\theta)$. Since the partition $\mathcal{P}$ has product structure, m^1 identifies some $\tilde{\theta} \in \mathcal{P}(\theta)$. To establish Step 2, we first show that for any agent i, $m_{i,j}^2 \sim_j \theta_j$ for all $j \neq i$. Suppose not, that is, there exists $j \neq i$ such that $m_{i,j}^2 \not\sim_j \theta_j$. We construct $\tilde{m}_i = \left(m_i^1, \left(\theta_{-i}, m_{i,i}^2\right)\right)$ which is identical to m_i except $(m_{i,j}^2)_{j \neq i}$. We claim that $\tilde{m}_i$ is strictly better than m_i against any $m_{-i} \in S_{-i}^{\infty}(\mathcal{M}, \theta)$.

Fix $m_{-i} \in S_{-i}^{\infty}(\mathcal{M}, \theta)$. Note that (m_i, m_{-i}) is "not" consistent since $m_i^2 \not\sim \theta$, which is implied by the hypothesis that $m_{i,j}^2 \not\sim_j \theta_j$ for some $j \neq i$.

We observe that $(\tilde{m}_i, m_{-i})$ and (m_i, m_{-i}) only differ in $(m_{i,j}^2)_{j \neq i}$ and $(\tilde{m}_{i,j}^2)_{j \neq i}$ within their second report. In terms of transfers incurred, $(\tilde{m}_i, m_{-i})$ avoids the penalty η due to $\tau_i^1(\cdot)$, while (m_i, m_{-i}) is penalized by η. In addition, (m_i, m_{-i}) suffers the penalty from $\tau_i^1(\cdot)$ and $(\tilde{m}_i, m_{-i})$ avoid this penalty. Hence, the transfer gain is at least η, which is larger than any possible utility loss from allocation. Hence, $\tilde{m}_i$ is a better reply than m_i against m_{-i}.

Finally, we claim that $m_i^2 \sim \theta$. By the previous argument, we know that in the second report, each agent $j \neq i$ announces agent i's type as $m_{j,i}^2 = \tilde{\theta}_i \sim \theta_i$. To avoid the penalty η due to $\tau_i^2(\cdot)$, agent i should announce m_i such that $m_{i,i}^2 \sim_i \theta_i$. Since the partition $\mathcal{P}$ has product structure, we conclude that $m_i^2 \sim \theta$. This completes the proof of Step 2.

Step 3: *For any agent* $i \in \mathcal{I}$ *and any* $m \in S^{\infty}(\mathcal{M}, \theta)$, *we have* $g(m) = f(\theta)$ *and* $\tau_i(m) = 0$.

By Steps 1 and 2, for any $m \in S^{\infty}(\mathcal{M}, \theta)$, we have that m^1 identifies some $\tilde{\theta} \in \mathcal{P}(\theta)$ and $\tilde{\theta} \sim m_i^2$ for every $i \in \mathcal{I}$. We thus conclude that for every $m \in S^{\infty}(\mathcal{M}, \theta)$, we have $e(m) = 0$ for any $i, j \in \mathcal{I}$ so that no transfer is invoked and $f(\tilde{\theta})$ is implemented. Again, since $\tilde{\theta} \in \mathcal{P}(\theta)$, it follows that $g(m) = f(\theta)$. This completes the proof of Step 3.

3.3 Continuous Implementation

Oury and Tercieux (2012) consider the following situation: the planner wants not only that there is an equilibrium that implements the SCF but also that the same equilibrium continues to implement the SCF in all the models *close* to his initial model. Hence, the SCF is *continuously* implementable. Oury and Tercieux (2012) obtain the following characterization of continuous implementation in their Theorem 4: an SCF is continuously implementable by a finite mechanism if it is exactly implementable in rationalizable strategies by a finite mechanism.[7] Since this result says nothing about the class of SCSs that are exactly implementable in rationalizable strategies by finite mechanisms, we view this as an important open question in the literature. We establish the following continuous implementation result which is a direct consequence of our Theorem 1 and Theorem 4 of Oury and Tercieux (2012).

Proposition 1 *If an SCF satisfies Maskin monotonicity*, it is continuously implementable by a finite mechanism.*

To the best of our knowledge, our Proposition 1 is the first result which continuously implements all Maskin monotonic* SCFs by a finite mechanism. The identified condition, Maskin monotonicity*, is strictly stronger than Maskin monotonicity, as we will show in Section 5. However, two caveats remain in relating Proposition 1 to Theorem 4 of Oury and Tercieux (2012). The first caveat is that we focus on complete information environments, whereas Oury and Tercieux deal with incomplete information environments where the baseline model can be an arbitrary finite type space. The second caveat is that we specialize in environments with lottery and transfer, whereas Oury and Tercieux impose no condition on the environments.

In incomplete information environments with lottery and transfer, Chen et al. (2019) made some progress in this direction. They show that any incentive compatible SCF is continuously implementable by a finite mechanism, provided that (i) we allow for arbitrarily small ex post transfers both on the equilibrium and off the equilibrium; (ii) each player knows his own payoff type; and (iii) agents' beliefs satisfy a generic correlation condition. In other words, under the three assumptions above, incentive compatibility is the only constraint for continuous implementation.

4 Responsive SCFs

Bergemann et al. (2011) introduce a condition on SCFs.

[7] In fact, assuming that sending messages is slightly costly, Oury and Tercieux also prove the converse: an SCF is continuously implementable by a finite mechanism only if it is rationalizably implementable by a finite mechanism.

Definition 4 *An SCF f is* ***responsive*** *if, for any pair of states $\theta, \theta' \in \Theta$, $f(\theta) = f(\theta') \Rightarrow \theta = \theta'$.*

Responsiveness requires that the SCF "responds" to a change in the state with a change in the social choice outcome. Observe that a responsive SCF that satisfies Maskin monotonicity must satisfy Maskin monotonicity*. Indeed, since $\mathcal{P}_f$ is the finest partition on Θ, for any two states θ and $\theta^{'}$, $\theta^{'} \in \mathcal{P}(\theta)$ is equivalent to $\theta^{'} \neq \theta$. We thus obtain the following corollary for the case of responsive SCFs.

Corollary 1 *Let f be a responsive SCF. Then, the SCF f is implementable in rationalizable strategies by a finite mechanism if and only if it satisfies Maskin monotonicity.*

Remark: Bergemann et al. (2011) prove that under the no-worst alternative condition (See Definition 4 of Bergemann et al. (2011), p. 1259), if there are at least three agents, f is responsive, and satisfies Maskin monotonicity, then it is implementable in rationalizable strategies by an infinite mechanism. Thus, we can handle the case of two agents unlike BMT.

In the case of responsive SCFs, strict Maskin monotonicity*, which is a necessary condition for rationalizable implementation, reduces to strict Maskin monotonicity. We formalize this result whose proof is omitted.

Corollary 2 *If an SCF f is responsive and satisfies strict Maskin monotonicity, it also satisfies strict Maskin monotonicity*.*

In what follows, we argue that the responsiveness of SCFs is tightly connected to the permissive result of virtual implementation by Abreu and Matsushima (1992), who show that when there are at least three agents, any SCF is *virtually* implementable in rationalizable strategies by a finite mechanism. An SCF f is said to be virtually implementable if, for any $\varepsilon \in (0, 1)$, the SCF f is exactly implementable with probability $1 - \varepsilon$. Fix an SCF f arbitrarily and let $\varepsilon \in (0, 1)$, which will be fixed later. Define $f^{\varepsilon} : \Theta \to \Delta(A)$ as follows: for any $\theta \in \Theta$,

$$f^{\varepsilon}(\theta) = \varepsilon y_i(\theta_i) + (1 - \varepsilon) f(\theta),$$

where $y_i(\theta_i)$ is the dictator lottery for type θ_i, as constructed in Lemma 1. Moreover, by adding small transfers to the dictator lotteries, we can make $y_i(\theta_i) \neq y_i(\theta_i^{'})$ whenever $\theta \neq \theta'$, without affecting the conclusion of Lemma 1 (i.e., (13) below). Therefore, $f^\varepsilon(\theta) \neq f^\varepsilon(\theta')$ whenever $\theta \neq \theta'$. In other words, we can make f^ε responsive. We now argue that f^ε is also Maskin monotonic.[8] Fix two states θ and θ' with $\theta \neq \theta'$ (and hence $f^\varepsilon(\theta) \neq f^\varepsilon(\theta')$). Since $\theta \neq \theta'$ and due to the construction of dictator lotteries, there must exist agent i for whom

$$u_i(y_i(\theta_i), \theta_i) > u_i(y_i(\theta_i'), \theta_i) \text{ and } u_i(y_i(\theta_i'), \theta_i') > u_i(y_i(\theta_i), \theta_i'). \tag{13}$$

We construct the following lottery $x(\theta', \theta_i) \in X$:

$$x(\theta', \theta_i) \equiv \varepsilon y_i(\theta_i) + (1-\varepsilon) f(\theta').$$

That is, $x(\theta', \theta_i)$ is constructed by replacing $y_i(\theta_i')$ in $f(\theta')$ with $y_i(\theta_i)$. By (13), we have

$$x(\theta', \theta_i) \in \mathcal{SL}_i\left(f^\varepsilon\left(\theta'\right), \theta_i'\right) \cap \mathcal{SU}_i\left(f^\varepsilon\left(\theta'\right), \theta_i\right).$$

This shows that f^ε satisfies strict Maskin monotonicity. By Theorem 1, we provide the following result without proof.

Corollary 3 *Any SCF f is virtually implementable in rationalizable strategies by a finite mechanism.*

Recall that our mechanism is different from that of Abreu and Matsushima (1992), who do not use transfers but rather introduce a domain restriction in the lottery space. AM's (1992) domain restriction requires that for every player i and state θ, there exist a pair of lotteries which are strictly ranked for player i and for which other players have the (weakly) opposite ranking.

5 Maskin Monotonicity and Maskin Monotonicity*

We construct an SCF which satisfies strict Maskin monotonicity but not strict Maskin monotonicity*.[9] This implies that rationalizable implementation is more restrictive than

[8] One additional property Abreu and Matsushima (1992) obtain in their mechanism is that they can make the size of transfers arbitrarily small. We discuss this below.

[9] This example is considered a two-agent version of the example in Appendix A of Jain (2017) which also accommodate the environments with lottery and transfers.

Nash implementation. Recall that in environments with transfers, strict Maskin monotonicity is equivalent to Maskin monotonicity and strict Maskin monotonicity* is also equivalent to Maskin monotonicity*. Let $A = \{a,b,c,d\}$, $\mathcal{I} = \{1,2\}$, $X = \Delta(A) \times \mathbb{R}^2$, and $\Theta = \{\alpha, \beta, \gamma, \delta\}$. The agents' utility functions are given in the two tables below. Consider the following SCF $f(\alpha) = f(\beta) = f(\gamma) = a$ and $f(\delta) = b$. For simplicity of notation, we write $\tilde{a} \in A$ for $(a,0,0) \in X$ which is a degenerate allocation with no transfer to any agent.

v_A	α	β	γ	δ
a	3	2	2	2
b	2	3	1	3
c	1	1	3	1
d	0	0	0	0

v_B	α	β	γ	δ
a	3	2	2	2
b	1	0	1	1
c	2	1	3	3
d	0	3	0	0

Claim 1 *For every agent i and $\theta \in \Theta$, $\mathcal{SL}_i(a,\theta) \subset \mathcal{L}_i(a,\alpha)$.*

Proof. Observe that for any agent, any $\tilde{a} \in A \setminus \{a\}$, and any $\theta \in \Theta$, the utility difference between a and $\tilde{a}$ is larger at α than at θ, that is,

$$v_i(a,\alpha) - v_i(\tilde{a},\alpha) \geq v_i(a,\theta) - v_i(\tilde{a},\theta).$$

Hence, for any $x \in X$, we have $u_i(a,\theta) - u_i(x,\theta) \geq 0$ whenever $u_i(a,\tilde{\theta}) - u_i(x,\tilde{\theta}) \geq 0$. ■

Claim 2 *The SCF f violates strict Maskin monotonicity*.*

Proof. Consider an arbitrary partition finer than $\mathcal{P}_f = \{\{\alpha,\beta,\gamma\},\{\delta\}\}$. Note that $\mathcal{P}(\delta) = \{\delta\}$ for any partition $\mathcal{P}$ finer than $\mathcal{P}_f$.

Case 1. $\alpha \in \mathcal{P}(\beta)$ and $\alpha \in \mathcal{P}(\gamma)$. In this case, $\mathcal{P} = \mathcal{P}_f$ and hence $\mathcal{P}(\alpha) = \{\alpha,\beta,\gamma\}$. Since $\mathcal{SL}_A(a,\beta) = \mathcal{SL}_A(a,\delta)$ and $\mathcal{SL}_B(a,\gamma) = \mathcal{SL}_B(a,\delta)$. Thus, $\mathcal{SL}_i(a,\mathcal{P}(\alpha)) \subset \mathcal{L}_i(a,\delta)$ but $f(\alpha) \neq f(\delta)$. Hence, f violates strict Maskin monotonicity* for such $\mathcal{P}$.

Case 2. $\alpha \notin \mathcal{P}(\beta)$ or $\alpha \notin \mathcal{P}(\gamma)$. We derive a contradiction for $\alpha \notin \mathcal{P}(\beta)$ and the argument for the case with $\alpha \notin \mathcal{P}(\gamma)$ is similar and so omitted. If $\alpha \notin \mathcal{P}(\beta)$, then by Claim 1, we have $\mathcal{SL}_i(a,\mathcal{P}(\beta)) \subset \mathcal{L}_i(a,\alpha)$. Then, f violates strict Maskin monotonicity* for $\mathcal{P}$ since $\mathcal{SL}_i(a,\mathcal{P}(\beta)) \subset \mathcal{L}_i(a,\alpha)$ and $\alpha \notin \mathcal{P}(\beta)$. ■

Claim 3 *The SCF f satisfies strict Maskin monotonicity.*

Proof. Indeed, observe that $b \in \mathcal{SL}_A(a,\alpha) \cap \mathcal{SU}_A(a,\delta)$, $c \in \mathcal{SL}_B(a,\beta) \cap \mathcal{SU}_B(a,\delta)$, $b \in \mathcal{SL}_A(a,\gamma) \cap \mathcal{SU}_A(a,\delta)$, $a \in \mathcal{SL}_A(b,\delta) \cap \mathcal{SU}_A(b,\alpha)$, $d \in \mathcal{SL}_B(b,\delta) \cap \mathcal{SU}_B(b,\beta)$, and $a \in \mathcal{SL}_A(b,\delta) \cap \mathcal{SL}_A(b,\gamma)$. ■

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